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P_5-factorization of complete bipartite graphs[☆]

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Abstract

A P_k-factor of complete bipartite graph $K_{m,n}$ is a spanning subgraph of $K_{m,n}$ such that every component is a path of length k. A P_k-factorization of $K_{m,n}$ is a set of edge-disjoint P_k-factors of $K_{m,n}$ which is a partition of the set of edges of $K_{m,n}$. When k is an even number, the spectrum problem for a P_k-factorization of $K_{m,n}$ has been completely solved. When k is an odd number, Ushio in 1993 proposed a conjecture. However, up to now we only know that Ushio Conjecture is true for $k=3$. In this paper we will show that Ushio Conjecture is true when $k=5$. That is, we shall prove that a necessary and sufficient condition for the existence of a P_5-factorization of $K_{m,n}$ is (1) $3n \geqslant 2m$, (2) $3m \geqslant 2n$, (3) $m+n \equiv 0 \pmod 5$, and (4) $5mn/[4(m+n)]$ is an integer.

Keywords: Complete bipartite graph; Path; Factorization

1. Introduction

Let P_k be the path on k vertices and $K_{m,n}$ be the complete bipartite graph with partite sets V_1 and V_2, where $|V_1|=m$ and $|V_2|=n$. A subgraph F of $K_{m,n}$ is called a spanning subgraph of $K_{m,n}$ if F contains all the vertices of $K_{m,n}$. A P_k-factor of $K_{m,n}$ is a spanning subgraph F of $K_{m,n}$ such that every component of F is a P_k and every pair of P_k's has no vertex in common. A P_k-factorization of $K_{m,n}$ is a set of edge-disjoint P_k-factors of $K_{m,n}$ which is a partition of the set of edges of $K_{m,n}$. In paper [6], the P_k-factorization of $K_{m,n}$ is defined as a resolvable $(m, n, k, 1)$ bipartite P_k-design. The graph $K_{m,n}$ is called P_k-factorizable whenever it has a P_k-factorization. For graph theoretical terms, see [4].

When k is an even number, the spectrum problem for a P_k-factorization of $K_{m,n}$ has been completely solved (see [3,6,8]). When k is an odd number, the spectrum problem for a P_k-factorization of $K_{m,n}$ seems to be much less tractable. Ushio in [5] gave a necessary and sufficient condition for existence of P_3-factorization of $K_{m,n}$. Some further work was done by Ushio and Tsuruno in [7], Du in [1,2], and Wang and Du in [9]. In paper [6], Ushio proposed the following conjecture [6, Conjecture 5.3].

Conjecture 1.1. Let m and n be positive integers and k be odd. Then $K_{m,n}$ has a P_k-factorization if and only if (1) $(k+1)n \geqslant (k-1)m$, (2) $(k+1)m \geqslant (k-1)n$, (3) $m+n \equiv 0 \pmod k$, and (4) $kmn/[(k-1)(m+n)]$ is an integer.

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However, up to now we only know that Ushio Conjecture is true for $k = 3$. In this paper we will show that Ushio Conjecture is true when $k = 5$. That is, we shall prove:

Theorem 1.2. *Let m and n be positive integers. Then $K_{m,n}$ has a P_5-factorization if and only if* (1) $3n \geqslant 2m$, (2) $3m \geqslant 2n$, (3) $m + n \equiv 0 \pmod 5$, *and* (4) $5mn/[4(m + n)]$ *is an integer.*

2. Proof of the main result

First, assume that a P_5-factorization of $K_{m,n}$ is given. Certain integers are defined as follows:
t = the number of copies of P_5 in any factor,
r = the number of P_5-factors in the factorization,
a = the number of copies of P_5 with its endpoints in Y in a particular P_5-factor (type M),
b = the number of copies of P_5 with its endpoints in X in a particular P_5-factor(type W),
c = the total number of copies of P_5 in the whole factorization.

Since any P_5-factor spans $K_{m,n}$,

$$t = \frac{m+n}{5}. \tag{2.1}$$

Every P_5-factor has $4t$ edges so that in a factorization $mn = 4rt = 4c$. Thus

$$r = \frac{5mn}{4(m+n)}. \tag{2.2}$$

By definition of a and b, we get $2a + 3b = m$ and $3a + 2b = n$. Hence

$$a = \frac{3n-2m}{5}, \tag{2.3}$$

$$b = \frac{3m-2n}{5}. \tag{2.4}$$

Since expressions (2.1)–(2.4) must be integers, we have the following necessary condition for the existence of a P_5-factorization of the complete bipartite graph $K_{m,n}$.

Lemma 2.1. *If $K_{m,n}$ has a P_5-factorization, then* (1) $3n \geqslant 2m$, (2) $3m \geqslant 2n$, (3) $m + n \equiv 0 \pmod 5$, *and* (4) $5mn/[4(m + n)]$ *is an integer.*

The remainder of this section is devoted to the proof of sufficiency theorem 1.2. For any two integers x and y, we use $gcd(x, y)$ to denote the greatest common divisor of x and y. The following lemma is obvious.

Lemma 2.2. *Let g, p and q be positive integers, if $gcd(p, q) = 1$, then*

$$gcd(pq, p + gq) = gcd(p, g).$$

We first prove the following result, which is used later in this paper.

Lemma 2.3. *If $K_{m,n}$ has a P_5-factorization, then $K_{sm,sn}$ has a P_5-factorization for every positive integer s.*

Proof. Let $\{F_i : 1 \leqslant i \leqslant s\}$ be a P_2-factorization of $K_{s,s}$ (whose existence, see [4]). For each $i \in \{1, 2, \ldots, s\}$, replace every edge of F_i by a $K_{m,n}$ to get a factor G_i of $K_{sm,sn}$ such that the graph G_i are pairwise edge-disjoint and their union is $K_{sm,sn}$. Since $K_{m,n}$ has a P_5-factorization, it is clear that the graph G_i, too, has a P_5-factorization. Consequently, $K_{sm,sn}$ has a P_5-factorization. This proves the theorem. □

Now we start to prove our main result Theorem 1.2. There are three cases to consider.

Case $3m = 2n$: In this case, combining conditions (4) in Theorem 1.2, m and n must satisfy $m = 4s$ and $n = 6s$ for some positive integer s. From Theorem 2.3, $K_{m,n}$ has a P_5-factorization, since $K_{4,6}$ has a P_5-factorization:

$$\begin{aligned}
&y_1x_1y_2x_2y_3, \quad y_4x_3y_5x_4y_6;\\
&y_3x_1y_4x_2y_5, \quad y_6x_3y_1x_4y_2;\\
&y_5x_1y_6x_2y_1, \quad y_2x_3y_3x_4y_4.
\end{aligned}$$

Case $2m = 3n$: Obviously, $K_{m,n}$ has a P_5-factorization.

Case $3m > 2n$ and $3n > 2m$: In this case, let $a=(3n-2m)/5$, $b=(3m-2n)/5$, $t=(m+n)/5$, and $r=5mn/[4(m+n)]$. Then from conditions (1)–(4) in Theorem 1.2, a, b, t, r are integers and $0 < a < m$ and $0 < b < n$. We have $2a+3b=m$ and $3a+2b=n$. Hence $r = 3(a+b)/2 + ab/[4(a+b)]$. Let $z = ab/[2(a+b)]$, which is a positive integer. And let $gcd(2a, 3b) = d$, $2a = dp$, $3b = dq$, where $gcd(p, q) = 1$. Then $z = dpq/[2(3p+2q)]$. These equalities imply the following equalities:

$$\begin{aligned}
d &= \frac{2(3p+2q)z}{pq},\\
m &= \frac{2(p+q)(3p+2q)z}{pq},\\
n &= \frac{(9p+4q)(3p+2q)z}{3pq},\\
r &= \frac{(p+q)(9p+4q)z}{2pq},\\
a &= \frac{p(3p+2q)z}{pq},\\
b &= \frac{2q(3p+2q)z}{3pq}.
\end{aligned}$$

Now we can establish the following lemma. The point of Lemma 2.4 is to discuss the details of spectrum of such factorization and reduce it to a number of base cases which are then solved in the later lemmas.

Lemma 2.4. (1) *If* $gcd(p, 4) = 1$ *and* $gcd(q, 9) = 1$, *then there is a positive integer* s *such that*

$$m = 12(p+q)(3p+2q)s, \quad n = 2(9p+4q)(3p+2q)s,$$

when $p + q \equiv 1 \pmod 2$; *or*

$$m = 6(p+q)(3p+2q)s, \quad n = (9p+4q)(3p+2q)s,$$

when $p + q \equiv 0 \pmod 2$.

(2) *If* $gcd(p, 4) = 1$ *and* $gcd(q, 9) = 3$, *let* $q = 3q'$. *Then there is a positive integer* s *such that*

$$m = 4(p+3q')(p+2q')s, \quad n = 2(3p+4q')(p+2q')s,$$

when $p + 3q' \equiv 1 \pmod 2$; *or*

$$m = 2(p+3q')(p+2q')s, \quad n = (3p+4q')(p+2q')s,$$

when $p + 3q' \equiv 0 \pmod 2$.

(3) *If* $gcd(p, 4) = 1$ *and* $gcd(q, 9) = 9$, *let* $q = 9q''$. *Then there is a positive integer* s *such that*

$$m = 4(p+9q'')(p+6q'')s, \quad n = 6(p+4q'')(p+6q'')s,$$

when $p + 9q'' \equiv 1 \pmod 2$; *or*

$$m = 2(p+9q'')(p+6q'')s, \quad n = 3(p+4q'')(p+6q'')s,$$

when $p + 9q'' \equiv 0 \pmod 2$.

(4) *If* $gcd(p,4)=2$ *and* $gcd(q,9)=1$, *let* $p=2p'$. *Then there is a positive integer s such that*

$$m=12(2p'+q)(3p'+q)s, \quad n=4(9p'+2q)(3p'+q)s.$$

(5) *If* $gcd(p,4)=2$ *and* $gcd(q,9)=3$, *let* $p=2p'$ *and* $q=3q'$. *Then there is a positive integer s such that*

$$m=4(2p'+3q')(p'+q')s, \quad n=4(3p'+2q')(p'+q')s.$$

(6) *If* $gcd(p,4)=2$ *and* $gcd(q,9)=9$, *let* $p=2p'$ *and* $q=9q''$. *Then there is a positive integer s such that*

$$m=4(2p'+9q'')(p'+3q'')s, \quad n=12(p'+2q'')(p'+3q'')s.$$

(7) *If* $gcd(p,4)=4$ *and* $gcd(q,9)=1$, *let* $p=4p''$. *Then there is a positive integer s such that*

$$m=6(4p''+q)(6p''+q)s, \quad n=4(9p''+q)(6p''+q)s,$$

when $9p''+q\equiv 1\ (\mathrm{mod}\ 2)$; *or*

$$m=3(4p''+q)(6p''+q)s, \quad n=2(9p''+q)(6p''+q)s,$$

when $9p''+q\equiv 0\ (\mathrm{mod}\ 2)$.

(8) *If* $gcd(p,4)=4$ *and* $gcd(q,9)=3$, *let* $p=4p''$ *and* $q=3q'$. *Then there is a positive integer s such that*

$$m=2(4p''+3q')(2p''+q')s, \quad n=4(3p''+q')(2p''+q')s,$$

when $3p''+q'\equiv 1\ (\mathrm{mod}\ 2)$; *or*

$$m=(4p''+3q')(2p''+q')s, \quad n=2(3p''+q')(2p''+q')s,$$

when $3p''+q'\equiv 0\ (\mathrm{mod}\ 2)$.

(9) *If* $gcd(p,4)=4$ *and* $gcd(q,9)=9$, *let* $p=4p''$ *and* $q=9q''$. *Then there is a positive integer s such that*

$$m=2(4p''+9q'')(2p''+3q'')s, \quad n=12(p''+q'')(2p''+3q'')s,$$

when $p''+q''\equiv 1\ (\mathrm{mod}\ 2)$; *or*

$$m=(4p''+9q'')(2p''+3q'')s, \quad n=6(p''+q'')(2p''+3q'')s,$$

when $p''+q''\equiv 0\ (\mathrm{mod}\ 2)$.

Proof. We assume that $gcd(p,q)=1$, $gcd(p,4)=1$ and $gcd(q,9)=1$ hold. Then $gcd(9p+4q,3)=gcd(3p+2q,3)=1$, $gcd(9p,4)=gcd(3p,2)=gcd(9p+4q,2)=1$ hold. By Lemma 2.2, we have $gcd(pq,9p+4q)=gcd(pq,3p+2q)=1$. Since $n=(9p+4q)(3p+2q)z/(3pq)$ is an integer, we see that $z/(3pq)$ must be an integer. It is easy to see $r=(p+q)(9p+4q)z/(2pq)$ is an integer too, then $z/(6pq)$ must be an integer when $p+q\equiv 1\ (\mathrm{mod}\ 2)$, or $z/(3pq)$ must be an integer when $p+q\equiv 0\ (\mathrm{mod}\ 2)$. So let $s=z/(6pq)$ when $p+q\equiv 1\ (\mathrm{mod}\ 2)$, or let $s=z/(3pq)$ when $p+q\equiv 0\ (\mathrm{mod}\ 2)$. Then the equalities in (1) hold.

The proof of the equalities in (2)–(3) and (7)–(9) are similar to (1).

We assume that $gcd(p,q)=1$, $gcd(p,4)=2$, $gcd(q,9)=1$ and $p=2p'$ hold. Then $gcd(p',q)=gcd(p',2)=gcd(q,2)=1$, $gcd(2p'+q,2)=gcd(9p'+2q,2)=1$ and $gcd(9p'+2q,3)=gcd(3p'+q,3)=1$ hold. By Lemma 2.2, we have $gcd(p'q,9p'+2q)=gcd(p'q,3p'+q)=1$. Since $n=2(9p'+2q)(3p'+q)z/(3p'q)$ is an integer, we see that $z/(3p'q)$ must be an integer. It is easy to see $r=(2p'+q)(9p'+2q)z/(2p'q)$ is an integer too, then $z/(6p'q)$ must be an integer. Let $s=z/(6p'q)$. Then the equalities in (4) hold.

The proof of the equalities in (5) and (6) are similar to (4).

This proves the lemma. □

From 9 parts in Lemma 2.4, there are 15 base cases to be considered. Notice m and n in (1) and (9), (2) and (8), (3) and (7), (4) and (6), are symmetric, respectively. So, for our main result, we only need 8 direct constructions. In the remainder of this paper, Lemmas 2.5 and 2.10 deal with the base case in (1) and (9) in Lemma 2.4, Lemmas 2.6 and

2.11 deal with the base case in (2) and (8), Lemmas 2.7 and 2.12 deal with the base case in (3) and (9), Lemma 2.8 deals with the base case in (4) and (6) and Lemma 2.9 deals with the base case in (5).

Lemma 2.5. *For any positive integers p and q, let* $m = 12(p+q)(3p+2q)$ *and* $n = 2(9p+4q)(3p+2q)$. *Then* $K_{m,n}$ *has a* P_5*-factorization.*

Proof. Let $r = 3(p+q)(9p+4q)$, $a = 6p(3p+2q)$, $b = 4q(3p+2q)$ and $t = 2(3p+2q)^2$. Let $r_1 = 3(p+q)$, $r_2 = 9p+4q$, $m_0 = m/r_1 = 4(3p+2q)$ and $n_0 = n/r_2 = 2(3p+2q)$. Let X and Y be two partite sets of $K_{m,n}$ and set

$$X = \{x_{i,j}: 1 \leqslant i \leqslant r_1, 1 \leqslant j \leqslant m_0\},$$

$$Y = \{y_{i,j}: 1 \leqslant i \leqslant r_2, 1 \leqslant j \leqslant n_0\}.$$

We remark in advance that the additions in the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo r_1 and r_2 in $\{1, 2, \ldots, r_1\}$ and $\{1, 2, \ldots, r_2\}$, respectively, and the additions in the second subscripts of $x_{i,j}$'s and $y_{i,j}$'s are taken modulo m_0 and n_0 in $\{1, 2, \ldots, m_0\}$ and $\{1, 2, \ldots, n_0\}$, respectively.

In the remainder of the proof, we construct a model of a P_5- factor of $K_{m,n}$, and then get $r = r_1 r_2$ edges-disjoint P_5-factors by rotational variants of this model, finally piece these factors together to form the required factorization.

For making a P_5-factor of $K_{m,n}$, we need $t = a + b = 2(3p+2q)^2$ vertex-disjoint copies of P_5. Among these copies, there are of $a = 6p(3p+2q)$ Type M P_5 and $b = 4q(3p+2q)$ Type W P_5, where Type M denotes the P_5 with its endpoints in Y and Type W with its endpoints in X.

Type M copies of P_5.

Notice $m_0 = 2n_0$; we let $E_0 = \{x_{1,1}y_{1,1}, x_{1,1}y_{2,2}, x_{1,1+m_0/2}y_{2,2}, x_{1,1+m_0/2}y_{3,3}\}$, which is a Type M P_5. For $1 \leqslant j \leqslant n_0$, it is easy to see that $E = \{x_{1,j}y_{1,j}, x_{1,j}y_{2,j+1}, x_{1,j+m_0/2}y_{2,j+1}, x_{1,j+m_0/2}y_{3,j+2}: 1 \leqslant j \leqslant n_0\}$ contains n_0 vertex-disjoint Type M copies.

For each $1 \leqslant j \leqslant n_0$ and each $0 \leqslant u, v \leqslant 1$, we may express E more conveniently as $E = \{x_{1,j+m_0u/2}y_{u+v+1,j+u+v}: 1 \leqslant j \leqslant n_0, 0 \leqslant u, v \leqslant 1\}$. For $1 \leqslant i \leqslant 3p$, let the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s in E add expression "i" and "$3(i-1)$", the second subscripts of $y_{i,j}$'s add expression "$2(i-1)$". We get all Type M copies.

For each $1 \leqslant i \leqslant 3p$, let

$$E_i = \{x_{i+1,j+m_0u/2}y_{3(i-1)+u+v+1,j+2(i-1)+u+v}: 1 \leqslant j \leqslant n_0, 0 \leqslant u, v \leqslant 1\}.$$

Each of E_i $(1 \leqslant i \leqslant 3p)$ consists of n_0 vertex-disjoint Type M copies. And $\bigcup_{1 \leqslant i \leqslant 3p} E_i$ contains $a = 6p(3p+2q) = 3pn_0$ vertex-disjoint Type M copies of P_5.

Type W copies of P_5.

Let $R_0 = \{x_{1,j}y_{1,j}, x_{2,j}y_{1,j+1}, x_{2,j}y_{2,j+2}, x_{3,j}y_{2,j+3}: 1 \leqslant j \leqslant n_0\}$. R_0 contains $4n_0$ edges, these edges can be partitioned into n_0 vertex-disjoint Type W copies of P_5. Similarly, $R_1 = \{x_{1,j+m_0/2}y_{3,j+1}, x_{2,j+m_0/2}y_{3,j+2}, x_{2,j+m_0/2}y_{4,j+3}, x_{3,j+m_0/2}y_{4,j+4}: 1 \leqslant j \leqslant n_0\}$ contains n_0 vertex-disjoint Type W copies, too.

Let $R = R_0 \cup R_1$, which contains $2n_0$ vertex-disjoint Type W copies. For $1 \leqslant u \leqslant 2$ and each $0 \leqslant v, w \leqslant 1$, we may express R more conveniently as $R = \{x_{u+v,j+m_0w/2}y_{2w+u,j+2u+w+v-2}: 1 \leqslant j \leqslant n_0, 1 \leqslant u \leqslant 2, 0 \leqslant v, w \leqslant 1\}$.

Finally, for $1 \leqslant i \leqslant q$, let the first subscripts of $x_{i,j}$'s and $y_{i,j}$'s in R add expression "$3p+3(i-1)$" and "$9p+4(i-1)$", the second subscripts of $y_{i,j}$'s add expression "$6p+4(i-1)$". We get all Type W copies.

For each $1 \leqslant i \leqslant q$, $1 \leqslant u \leqslant 2$ and each $0 \leqslant v, w \leqslant 1$, let

$$E_{3p+i} = \{x_{3p+3(i-1)+u+v,j+m_0w/2}y_{9p+4(i-1)+2w+u,6p+j+4(i-1)+2u+v+w-2}: 1 \leqslant j \leqslant n_0, 1 \leqslant u \leqslant 2, 0 \leqslant v, w \leqslant 1\}.$$

By computation, each of E_{3p+i} $(1 \leqslant i \leqslant q)$ consists of $2n_0$ vertex-disjoint Type W copies. And $\bigcup_{1 \leqslant i \leqslant q} E_{3p+i}$ contains $b = 4q(3p+2q) = 2qn_0$ vertex-disjoint Type W copies of P_5.

Let $F = \bigcup_{1 \leqslant i \leqslant 3p+q} E_i$. Obviously, F contains $t = a+b = 2(3p+2q)^2 = (3p+2q)n_0$ vertex-disjoint and edges-disjoint P_5 components, and spans $K_{m,n}$. Then the graph F is a P_5-factor of $K_{m,n}$. Further, in $\bigcup_{1 \leqslant i \leqslant 3p} E_i$, for $1 \leqslant j \leqslant n_0$, each of the second subscripts of $x_{i,j}$'s "j" meets the second subscripts of $y_{i,j}$'s from "j" to "$6p-1+j$" and "$j+m_0/2$" meets the second subscripts of $y_{i,j}$'s from "$1+j$" to "$6p+j$" once and only once. And in $\bigcup_{3p+1 \leqslant i \leqslant 3p+q} E_i$, for $1 \leqslant j \leqslant n_0$, each of the second subscripts of $x_{i,j}$'s "j" meets the second subscripts of $y_{i,j}$'s from "$6p+j$" to "$6p+4q-1+j = n_0+j-1$" and "$j+m_0/2$" meets the second subscripts of $y_{i,j}$'s from "$6p+1+j$" to

"$6p+4q+j=n_0+j$" once and only once. Then in the graph $F=\bigcup_{1\leqslant i\leqslant 3p+q} E_i$, each of the second subscripts of $x_{i,j}$'s meets each of the second subscripts of $y_{i,j}$'s once and only once.

Define a bijection σ from $X\cup Y$ onto $X\cup Y$ in such a way that $\sigma(x_{i,j})=x_{i+1,j}$ and $\sigma(y_{i,j})=y_{i+1,j}$. For each $i\in\{1,2,\ldots,r_1\}$ and each $j\in\{1,2,\ldots,r_2\}$, let

$$F_{\mu,v}=\{\sigma^{\mu}(x)\sigma^{v}(y)\colon x\in X,\ y\in Y,\ xy\in F\}.$$

It is shown that the graphs $F_{\mu,v}$ $(1\leqslant\mu\leqslant r_1,\ 1\leqslant v\leqslant r_2)$ are edge-disjoint P_5-factors of $K_{m,n}$ and their union is $K_{m,n}$. Thus, $\{F_{\mu,v}\colon 1\leqslant\mu\leqslant r_1,\ 1\leqslant v\leqslant r_2\}$ is a P_5-factorization of $K_{m,n}$. This proves the lemma. $\square$

The proof of the following lemmas is similar to Lemma 2.5, so we only give the representations of X, Y, E_i and E_{p+i} or (E_{3p+i}).

Lemma 2.6. *For any positive integers p and q, let $m=4(p+3q)(p+2q)$ and $n=2(3p+4q)(p+2q)$. Then $K_{m,n}$ has a P_5-factorization.*

Proof. Let $r=(p+3q)(3p+4q)$, $a=2p(p+2q)$, $b=4q(p+2q)$ and $t=2(p+2q)^2$. Let $r_1=p+3q$, $r_2=3p+4q$, $m_0=m/r_1=4(p+2q)$ and $n_0=n/r_2=2(p+2q)$, and

$$X=\{x_{i,j}\colon 1\leqslant i\leqslant r_1,\ 1\leqslant j\leqslant m_0\},$$
$$Y=\{y_{i,j}\colon 1\leqslant i\leqslant r_2,\ 1\leqslant j\leqslant n_0\}.$$

For each $1\leqslant i\leqslant p$, $0\leqslant u\leqslant 1$, and each $1\leqslant v\leqslant 2$, let

$$E_i=\{x_{i,m_0u/2+j}\,y_{3(i-1)+u+v,j+2(i-1)+v}\colon 1\leqslant j\leqslant n_0,\ 0\leqslant u\leqslant 1,\ 1\leqslant v\leqslant 2\}.$$

For each $1\leqslant i\leqslant q$, $1\leqslant u\leqslant 2$ and each $0\leqslant v,\ w\leqslant 1$, let

$$E_{p+i}=\{x_{p+3(i-1)+u+v,m_0w/2+j}\,y_{3p+4(i-1)+2u-1+w,2p+2u+v+4(i-1)+j-1}\colon 1\leqslant j\leqslant n_0,\ 1\leqslant u\leqslant 2,\ 0\leqslant v,\ w\leqslant 1\}.\quad\square$$

Lemma 2.7. *For any positive integers p and q, let $m=4(p+9q)(p+6q)$ and $n=6(p+4q)(p+6q)$. Then $K_{m,n}$ has a P_5-factorization.*

Proof. Let $r=3(p+9q)(p+4q)$, $a=2p(p+6q)$, $b=12q(p+6q)$ and $t=2(p+6q)^2$. Let $r_1=p+9q$, $r_2=3(p+4q)$, $m_0=m/r_1=4(p+6q)$ and $n_0=n/r_2=2(p+6q)$, and

$$X=\{x_{i,j}\colon 1\leqslant i\leqslant r_1,\ 1\leqslant j\leqslant m_0\},$$
$$Y=\{y_{i,j}\colon 1\leqslant i\leqslant r_2,\ 1\leqslant j\leqslant n_0\}.$$

For each $1\leqslant i\leqslant p$, $0\leqslant u\leqslant 1$, and each $1\leqslant v\leqslant 2$, let

$$E_i=\{x_{i,m_0u/2+j}\,y_{3(i-1)+u+v,j+2(i-1)+v}\ \colon 1\leqslant j\leqslant n_0,\ 0\leqslant u\leqslant 1,\ 1\leqslant v\leqslant 2\}.$$

For each $1\leqslant i\leqslant 3q$, $1\leqslant u\leqslant 2$, and each $0\leqslant v,\ w\leqslant 1$, let

$$E_{p+i}=\{x_{3(i-1)+p+u+v,m_0w/2+j}\,y_{4(i-1)+3p+2u+w-1,2p+j+4(i-1)+2u+v-1}\colon 1\leqslant j\leqslant n_0,\ 1\leqslant u\leqslant 2,\ 0\leqslant v,\ w\leqslant 1\}.\quad\square$$

Lemma 2.8. *For any positive integers p and q, let $m=12(2p+q)(3p+q)$ and $n=4(9p+2q)(3p+q)$. Then $K_{m,n}$ has a P_5 -factorization.*

Proof. Let $r=3(2p+q)(9p+2q)$, $a=12p(3p+q)$, $b=4q(3p+q)$ and $t=4(3p+q)^2$. Let $r_1=3(2p+q)$, $r_2=9p+2q$, $m_0=m/r_1=4(3p+q)$ and $n_0=n/r_2=4(3p+q)$, and

$$X=\{x_{i,j}\colon 1\leqslant i\leqslant r_1,\ 1\leqslant j\leqslant m_0\},$$
$$Y=\{y_{i,j}\colon 1\leqslant i\leqslant r_2,\ 1\leqslant j\leqslant n_0\}.$$

For each $1 \leqslant i \leqslant 3p$, $1 \leqslant u \leqslant 2$, and each $0 \leqslant v \leqslant 1$, let

$$E_i = \{x_{2(i-1)+u,j}\, y_{3(i-1)+u+v,j+4(i-1)+2u+v-1} \colon 1 \leqslant j \leqslant n_0,\ 1 \leqslant u \leqslant 2,\ 0 \leqslant v \leqslant 1\}.$$

For each $1 \leqslant i \leqslant q$, $1 \leqslant u \leqslant 2$, and each $0 \leqslant v \leqslant 1$, let

$$E_{3p+i} = \{x_{3(i-1)+6p+u+v,j}\, y_{2(i-1)+9p+u,j+12p+4(i-1)+2u+v-1} \colon 1 \leqslant j \leqslant n_0,\ 1 \leqslant u \leqslant 2,\ 0 \leqslant v \leqslant 1\}. \quad \square$$

Lemma 2.9. *For any positive integers p and q, let $m = 4(2p + 3q)(p + q)$ and $n = 4(3p + 2q)(p + q)$. Then $K_{m,n}$ has a P_5 -factorization.*

Proof. Let $r = (2p + 3q)(3p + 2q)$, $a = 4p(p + q)$, $b = 4q(p + q)$ and $t = 4(p + q)^2$. Let $r_1 = 2p + 3q$, $r_2 = 3p + 2q$, $m_0 = m/r_1 = 4(p + q)$ and $n_0 = n/r_2 = 4(p + q)$, and

$$X = \{x_{i,j} \colon 1 \leqslant i \leqslant r_1,\ 1 \leqslant j \leqslant m_0\},$$
$$Y = \{y_{i,j} \colon 1 \leqslant i \leqslant r_2,\ 1 \leqslant j \leqslant n_0\}.$$

For each $1 \leqslant i \leqslant p$, $1 \leqslant u \leqslant 2$, and each $0 \leqslant v \leqslant 1$, let

$$E_i = \{x_{2(i-1)+u,j}\, y_{3(i-1)+u+v,j+4(i-1)+2(u-1)+v+1} \colon 1 \leqslant j \leqslant n_0,\ 1 \leqslant u \leqslant 2,\ 0 \leqslant v \leqslant 1\}.$$

For each $1 \leqslant i \leqslant q$, $1 \leqslant u \leqslant 2$, and each $0 \leqslant v \leqslant 1$, let

$$E_{p+i} = \{x_{3(i-1)+2p+u+v,j}\, y_{2(i-1)+3p+u,j+4p+4(i-1)+2(u-1)+v+1} \colon 1 \leqslant j \leqslant n_0,\ 1 \leqslant u \leqslant 2,\ 0 \leqslant v \leqslant 1\}. \quad \square$$

In the following direct constructions, we use $\lceil x \rceil$ to denote the least integer not less than x and $\lfloor x \rfloor$ the largest integer not exceeding x.

Lemma 2.10. *For any coprime pair of positive integers p and q, in which p is an odd number and $p + q \equiv 0 \pmod 2$, let $m = 6(p + q)(3p + 2q)$ and $n = (9p + 4q)(3p + 2q)$. Then $K_{m,n}$ has a P_5 -factorization.*

Proof. Let $r = 3(p + q)(9p + 4q)/2$, $a = 3p(3p + 2q)$, $b = 2q(3p + 2q)$ and $t = (3p + 2q)^2$. Let $r_1 = 3(p + q)/2$, $r_2 = 9p + 4q$, $m_0 = m/r_1 = 4(3p + 2q)$ *and* $n_0 = n/r_2 = 3p + 2q$, and

$$X = \{x_{i,j} \colon 1 \leqslant i \leqslant r_1,\ 1 \leqslant j \leqslant m_0\},$$
$$Y = \{y_{i,j} \colon 1 \leqslant i \leqslant r_2,\ 1 \leqslant j \leqslant n_0\}.$$

For each $1 \leqslant i \leqslant 3p$, and each $0 \leqslant u, v \leqslant 1$, let

$$E_i = \{x_{\lceil i/2 \rceil, j+m_0u/4+m_0[1-(\lceil i/2 \rceil - \lfloor i/2 \rfloor)]/2}\, y_{3(i-1)+u+v+1, j+2\lceil i/2 \rceil -1+v} \colon 1 \leqslant j \leqslant n_0,\ 0 \leqslant u, v \leqslant 1\}.$$

For each $0 \leqslant u, v \leqslant 1$, let

$$\begin{aligned} E_{3p+1} = \{ & x_{\lceil 3p/2 \rceil +1, j+m_0u/2+m_0v/4}\, y_{9p+2u+v+1, j+2\lceil 3p/2 \rceil +1}, \\ & x_{\lceil 3p/2 \rceil +(1-u), j+m_0v/4+m_0/2}\, y_{9p+2u+v+1, j+2\lceil 3p/2 \rceil -1+u} \colon 1 \leqslant j \leqslant n_0,\ 0 \leqslant u, v \leqslant 1\}. \end{aligned}$$

For each $2 \leqslant i \leqslant q$, $0 \leqslant u, v \leqslant 1$, let

$$\begin{aligned} E_{3p+i} = \{ & x_{\lceil 3p/2 \rceil + \lceil 3i/2 \rceil -1, j+m_0u/2+m_0v/4}\, y_{9p+4(i-1)+2u+v+1, j+3p+2i-1}, \\ & x_{\lceil 3p/2 \rceil + \lceil 3i/2 \rceil -1+[1-(\lceil i/2 \rceil - \lfloor i/2 \rfloor)]u-(\lceil i/2 \rceil - \lfloor i/2 \rfloor)(1-u), j+m_0(1-u)/2+m_0v/4} \\ & y_{9p+4(i-1)+2u+v+1, j+3p+2i} \colon 1 \leqslant j \leqslant n_0,\ 0 \leqslant u, v \leqslant 1\}. \quad \square \end{aligned}$$

Lemma 2.11. *For any coprime pair of positive integers p and q, in which p is an odd number and $p + 3q \equiv 0 \pmod 2$, let $m = 2(p + 3q)(p + 2q)$ and $n = (3p + 4q)(p + 2q)$. Then $K_{m,n}$ has a P_5-factorization.*

Proof. Let $r=(p+3q)(3p+4q)/2$, $a=p(p+2q)$, $b=2q(p+2q)$ and $t=(p+2q)^2$. Let $r_1=(p+3q)/2$, $r_2=3p+4q$, $m_0=m/r_1=4(p+2q)$ and $n_0=n/r_2=p+2q$, and

$$X=\{x_{i,j}\colon 1\leqslant i\leqslant r_1,\ 1\leqslant j\leqslant m_0\},$$

$$Y=\{y_{i,j}\colon 1\leqslant i\leqslant r_2,\ 1\leqslant j\leqslant n_0\}.$$

For each $1\leqslant i\leqslant p$, and each $0\leqslant u,\ v\leqslant 1$, let

$$E_i=\{x_{\lceil i/2\rceil,j+m_0u/4+m_0[1-(\lceil i/2\rceil-\lfloor i/2\rfloor)]/2}y_{3(i-1)+u+v+1,j+2\lceil i/2\rceil-1+v}\colon 1\leqslant j\leqslant n_0,\ 0\leqslant u,\ v\leqslant 1\}.$$

For each $0\leqslant u,\ v\leqslant 1$, let

$$E_{p+1}=\{x_{\lceil p/2\rceil+1,j+m_0u/2+m_0v/4}y_{3p+2u+v+1,j+2\lceil p/2\rceil+1},$$
$$x_{\lceil p/2\rceil+(1-u),j+m_0v/4+m_0/2}y_{3p+2u+v+1,j+2\lceil p/2\rceil-1+u}\colon 1\leqslant j\leqslant n_0,\ 0\leqslant u,\ v\leqslant 1\}.$$

For each $2\leqslant i\leqslant q$, $0\leqslant u,\ v\leqslant 1$, let

$$E_{p+i}=\{x_{\lceil p/2\rceil+\lceil 3i/2\rceil-1,j+m_0u/2+m_0v/4}y_{3p+4(i-1)+2u+v+1,j+p+2i-1},$$
$$x_{\lceil p/2\rceil+\lceil 3i/2\rceil-1+[1-(\lceil i/2\rceil-\lfloor i/2\rfloor)]u-(\lceil i/2\rceil-\lfloor i/2\rfloor)(1-u),j+m_0(1-u)/2+m_0v/4}$$
$$y_{3p+4(i-1)+2u+v+1,j+p+2i}\colon 1\leqslant j\leqslant n_0,\ 0\leqslant u,\ v\leqslant 1\}.\quad\square$$

Lemma 2.12. *For any coprime pair of positive integers p and q, in which p is an odd number and* $p+9q\equiv 0\pmod 2$, *let* $m=2(p+9q)(p+6q)$ *and* $n=3(p+4q)(p+6q)$. *Then* $K_{m,n}$ *has a* P_5*-factorization.*

Proof. Let $r=3(p+9q)(p+4q)/2$, $a=p(p+6q)$, $b=6q(p+6q)$ and $t=(p+6q)^2$. Let $r_1=(p+9q)/2$, $r_2=3(p+4q)$, $m_0=m/r_1=4(p+6q)$ and $n_0=n/r_2=p+6q$, and

$$X=\{x_{i,j}\colon 1\leqslant i\leqslant r_1,\ 1\leqslant j\leqslant m_0\},$$

$$Y=\{y_{i,j}\colon 1\leqslant i\leqslant r_2,\ 1\leqslant j\leqslant n_0\}.$$

For each $1\leqslant i\leqslant p$, and each $0\leqslant u,\ v\leqslant 1$, let

$$E_i=\{x_{\lceil i/2\rceil,j+m_0u/4+m_0[1-(\lceil i/2\rceil-\lfloor i/2\rfloor)]/2}y_{3(i-1)+u+v+1,j+2\lceil i/2\rceil-1+v}\colon 1\leqslant j\leqslant n_0,\ 0\leqslant u,\ v\leqslant 1\}.$$

For each $0\leqslant u,\ v\leqslant 1$, let

$$E_{p+1}=\{x_{\lceil p/2\rceil+1,j+m_0u/2+m_0v/4}y_{3p+2u+v+1,j+2\lceil p/2\rceil+1},$$
$$x_{\lceil p/2\rceil+(1-u),j+m_0v/4+m_0/2}y_{3p+2u+v+1,j+2\lceil p/2\rceil-1+u}$$
$$\colon 1\leqslant j\leqslant n_0,\ 0\leqslant u,\ v\leqslant 1\}.$$

For each $2\leqslant i\leqslant 3q$, $0\leqslant u,\ v\leqslant 1$, let

$$E_{p+i}=\{x_{\lceil p/2\rceil+\lceil 3i/2\rceil-1,j+m_0u/2+m_0v/4}y_{3p+4(i-1)+2u+v+1,j+p+2i-1},$$
$$x_{\lceil p/2\rceil+\lceil 3i/2\rceil-1+[1-(\lceil i/2\rceil-\lfloor i/2\rfloor)]u-(\lceil i/2\rceil-\lfloor i/2\rfloor)(1-u),j+m_0(1-u)/2+m_0v/4}$$
$$y_{3p+4(i-1)+2u+v+1,j+p+2i}\colon 1\leqslant j\leqslant n_0,\ 0\leqslant u,\ v\leqslant 1\}.\quad\square$$

By applying Lemma 2.3 with Lemmas 2.4, 2.5, 2.6, 2.7,2.8, 2.9, 2.10, 2.11, 2.12 it can be seen that when the parameters m and n satisfy conditions (1)–(4) in Theorem 1.2, the graph $K_{m,n}$ has a P_5-factorization.

Theorem 2.13. *For any positive integer m and n, if* (1) $3n\geqslant 2m$, (2) $3m\geqslant 2n$, (3) $m+n\equiv 0\pmod 5$, *and* (4) $5mn/[4(m+n)]$ *is an integer, then* $K_{m,n}$ *has a* P_5*-factorization.*

Proof. Applying Lemmas 2.3 with Lemmas 2.5–2.12, we can obtain the result.

Proof of Theorem 1.2. Combining Lemma 2.1 and Theorem 2.13, we complete the proof of Theorem 1.2. $\square$

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